# Analytical Solution of Laplace's Equation via Separation of Variables

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**DOI: https://doi.org/10.5281/zenodo.15366723**

**Published Date: 08-May-2025**

*Abstract:* **Laplace's equation is a pivotal partial differential equation (PDE) in mathematical physics, with applications spanning electrostatics, steady-state heat transfer, fluid mechanics, and gravitational fields. It models systems in equilibrium, where quantities like temperature or electric potential remain constant over time in the absence of sources. In two dimensions, Laplace's equation is written as:**

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

**where $u(x, y)$ denotes the potential at coordinates $(x, y)$. In three dimensions, it includes an additional term for the $z$-coordinate:**

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

**To solve this equation analytically, appropriate boundary conditions must be defined, such as Dirichlet (prescribed values) or Neumann (prescribed derivatives) conditions. Among the most effective methods for tackling Laplace's equation in regular geometries is separation of variables, which transforms the PDE into simpler ordinary differential equations (ODEs). This article delves into the separation of variables technique, focusing on its application to a two-dimensional Laplace's equation in a rectangular domain, with a detailed derivation, key assumptions, and a practical example.**

*Keywords:* **Laplace's equation, partial differential equation (PDE), mathematical physics.**

## 1. PHYSICAL SIGNIFICANCE OF LAPLACE'S EQUATION

Laplace's equation, an elliptic PDE, governs harmonic functions—smooth functions satisfying$\nabla^2 u = 0$. It describes steady-state conditions where a system has reached a balance. For instance:

- In **electrostatics**, $u$ represents the electric potential in a charge-free region.
- In **steady-state heat conduction**, $u$ denotes the temperature in a medium without heat sources.
- In **fluid dynamics**, $u$ may describe the potential for irrotational flow.

Boundary conditions are critical for a well-posed problem. For a rectangle defined by$0 \leq x \leq a, 0 \leq y \leq b$ , possible conditions include::

- **Dirichlet:** $u = f(x, y)$ on the boundaries.
- **Neumann:** $\frac{\partial u}{\partial n} = g(x, y)$ (normal derivative).
- **Mixed:** A blend of Dirichlet and Neumann conditions.

The separation of variables method excels in domains with regular shapes and homogeneous boundary conditions, such as zero potential on certain boundaries..

## 2. OVERVIEW OF THE SEPARATION OF VARIABLES TECHNIQUE

The separation of variables approach assumes that the solution$u(x, y)$ can be expressed as a product of functions, each depending on a single variable:

$$u(x, y) = X(x)Y(y).$$

This factorization allows the PDE to be split into two independent ODEs, one for each spatial coordinate. By substituting this form into Laplace's equation and rearranging, we obtain equations that can be solved separately, with their solutions combined to meet the boundary conditions. The method leverages the linearity of the equation and is particularly suited for problems with homogeneous or partially homogeneous boundaries.**Step-by-Step**

**Detailed Derivation**

Consider Laplace's equation in a rectangular domain $0 \leq x \leq a, 0 \leq y \leq b$, with the following boundary conditions:

- $u(0, y) = 0, u(a, y) = 0$
- $u(x, 0) = 0, u(x, b) = f(x)$

This setup represents a typical Dirichlet problem.

- **Insert the Product Solution:**

Assume $u(x, y) = X(x)Y(y)$ and substitute into:

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$

Compute the second partial derivatives:

- $\frac{\partial^2 u}{\partial x^2} = \frac{d^2 X}{dx^2} Y(y),$
- $\frac{\partial^2 u}{\partial y^2} = X(x) \frac{d^2 y}{dy^2},$

Substituting yields:

$$\frac{d^2 X}{dx^2} Y(y) + X(x) \frac{d^2 y}{dy^2} = 0$$

- **Isolate Variables:**

Divide both sides by $X(x)\,Y(y)$, assuming $X \neq 0$ and $Y \neq 0$:

$$\frac{1}{X(x)} \frac{d^2 X}{dx^2} + \frac{1}{Y(y)} \frac{d^2 y}{dy^2} = 0$$

This implies:

$\frac{1}{X(x)} \frac{d^2 X}{dx^2} = -\frac{1}{Y(y)} \frac{d^2 y}{dy^2}.$

Since the left side depends only on (x) and the right side only on (y), both must equal a constant, say $\lambda$:

$$\frac{1}{X(x)} \frac{d^2 X}{dx^2} = \lambda, \qquad -\frac{1}{Y(y)} \frac{d^2 y}{dy^2} = \lambda$$

This results in two ODEs:

- $X''(x) - \lambda X(x) = 0,$
- $Y''(y) + \lambda Y(y) = 0.$

The choice of $\lambda$ affects the solution form. To align with the homogeneous boundary conditions, we assume $\lambda > 0$.

- **Solve the (X)-Equation:**

For the $X(x)$ – equation:

$$X''(x) - \lambda X(x) = 0.$$

Let $\lambda = -\mu^2$ (where $\mu > 0$) to obtain oscillatory solutions:

$$X''(x) + \mu^2 X(x) = 0.$$

The general solution is:

$$X(x) = Acos(\mu x) + Bsin(\mu x).$$

Apply the boundary conditions:

- $u(0, y) = X(0)Y(y) = 0 \Longrightarrow X(0) = 0$:

$$X(0) = Acos(0) + Bsin(0) = A = 0 \Longrightarrow A = 0.$$

So, $X(x) = Bsin(\mu x)$.

- $u(a, y) = X(a)Y(y) = 0 \Longrightarrow X(a) = 0$:

$$X(a) = Bsin(\mu a) = 0.$$

For a non-trivial solution ($B \neq 0$):

$$sin(\mu a) = 0 \Longrightarrow \mu a = n\pi, n = 1,2,3, \ldots.$$

So:

$$\mu_n = \frac{n\pi}{a}, \lambda_n = -\mu_n^2 = -(\frac{n\pi}{a})^2.$$

The eigenfunctions are:

$$X_n(x) = B_n sin(\frac{n\pi x}{a}).$$

- **Solve the (Y)-Equation:**

The $Y$-equation is:

$$Y''(y) + \lambda_n Y(y) = 0.$$

- Substitute $\lambda_n = -(\frac{n\pi}{a})^2$:

$$Y''(y) - (\frac{n\pi}{a})^2 Y(y) = 0.$$

The general solution is:

$$Y_n(y) = C_n e^{\frac{n\pi}{a}} + D_n e^{-\frac{n\pi}{a}}.$$

Alternatively, use hyperbolic functions:

$$Y_n(y) = E_n sinh(\frac{n\pi y}{a}) + F_n cosh(\frac{n\pi y}{a}).$$

- **Apply the Bottom Boundary Condition:**

The solution for each (n) is:

$$u_n(x, y) = X_n(x)Y_n(y) = sin(\frac{n\pi x}{a})[E_n sinh(\frac{n\pi y}{a}) + F_n cosh(\frac{n\pi y}{a})].$$

Apply $u(x, 0) = 0 \Longrightarrow u_n(x, 0) = 0$:

- $u_n(x, 0) = \sin\left(\frac{n\pi x}{a}\right)[E_n \sinh(0) + F_n \cosh(0)] = \sin\left(\frac{n\pi x}{a}\right).$

Thus, $F_n = 0$, and:

$$u_n(x,y) = E_n si\,n\left(\frac{n\pi x}{a}\right) sinh(\frac{n\pi y}{a}).$$

- **From the General Solution:**

The general solution is a sum over all modes:

$$u(x,y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right) sinh(\frac{n\pi y}{a}),$$

where $A_n = E_n$.

- **Apply the Top Boundary Condition:**

At $y = b, u(x,b) = f(x)$:

$$u(x,b) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right) sinh(\frac{n\pi b}{a}) = f(x).$$

This is a Fourier sine series. The coefficients are found as:

$$A_n sinh(\frac{n\pi x}{a}) = \frac{2}{a}\int_0^a f(x)\sin\left(\frac{n\pi x}{a}\right) dx.$$

Thus:

$$A_n = \frac{2}{a sinh(\frac{n\pi b}{a})}\int_0^a f(x)\sin\left(\frac{n\pi x}{a}\right) dx.$$

## 3. PRACTICAL EXAMPLE: POTENTIAL IN A RECTANGULAR REGION

Consider a rectangular domain $0 \le x \le 1, 0 \le y \le 1$ with:

- $u(0,y) = 0, u(1,y) = 0,$
- $u(x,0) = 0, u(x,1) = sin(\pi x).$

This scenario might model the electric potential in a capacitor with a sinusoidal potential on the top edge. The solution is:

$$u(x,y) = \sum_{n=1}^{\infty} A_n \sin(n\pi x)\, sinh(n\pi y)$$

At $y = 1$:

$$\sum_{n=1}^{\infty} A_n \sin(n\pi x)\, sin\,h(n\pi) = \sin(\pi x)$$

The Fourier series has only one term $n = 1$:

$$A_1 sinh(\pi) sin(\pi x) = sin(\pi x) \Rightarrow A_1 = \frac{1}{sinh(\pi)}, A_n = 0\; for\; n \neq 1.$$

Thus:

$$u(x,y) = \frac{sin(\pi x) sinh(\pi y)}{sinh(\pi)}.$$

This solution describes a potential that varies sinusoidally in $x$ and grows hyperbolically in $y$, consistent with the boundary conditions.

## 4. STRENGTHS AND CONSTRAINTS

The separation of variables method is robust for Laplace's equation in regular domains with homogeneous boundary conditions, leveraging orthogonal functions like sines. However, it faces challenges:

- **Non-Homogeneous Conditions:** Requires additional series or transformations.
- **Complex Geometries:** Less effective for irregular domains, where numerical methods or conformal mapping are better suited.
- **Alternative Coordinates:** In polar or spherical systems, the method involves Bessel or Legendre functions, increasing complexity.

## 5. OTHER SOLUTION TECHNIQUES

Alternative approaches include:

- **Fourier Transforms:** Ideal for unbounded domains.
- **Conformal Mapping:** Useful for $2D$ irregular boundaries.
- **Green's Functions:** Builds solutions using fundamental solutions.
- **Numerical Methods:** Finite difference or finite element methods for complex or non-linear problems.

## 6. CONCLUSION

The separation of variables method offers a systematic way to solve Laplace's equation, converting a PDE into ODEs solvable via Fourier series and hyperbolic functions. This approach yields physically meaningful solutions for steady-state problems in regular domains, with applications in electrostatics, heat transfer, and fluid mechanics. Mastering this technique equips researchers to address a wide range of equilibrium phenomena.

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